Thermal ratchets driven by Poissonian white shot noise

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We investigate the overdamped transport of Brownian particles that are placed in spatially periodic potentials (without and *with* reflection symmetry) that are subjected to *both* Poissonian white shot noise and thermal, i.e., Gaussian, white equilibrium fluctuations. The probability current of the output process, which is shown to obey a second-order ordinary differential equation, is analyzed. The limit of strong Poissonian white shot noise is studied analytically; the resulting current is given in closed form in terms of two quadratures. For general forms of the periodic potential we present asymptotic expansions in terms of the ratio between the thermal and the shot noise intensity. Analytic results are presented for the class of piecewise linear, sawtoothlike ratchet potentials. Under specific conditions, the current exhibits a distinctive *nonmonotonic* dependence on such parameters as temperature and/or asymmetry of the periodic potential. [S1063-651X(97)06104-7]

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I. INTRODUCTION

A variety of phenomena in nature are based on transport of mass and energy. One can distinguish, at the macroscopic level, the convective and diffusive character of transport. The former is identified with systematic or directed motion. The latter is a result of random collisions. Recently, it has been realized that directed motion can be induced by nonthermal fluctuations acting in the absence of any gradient fields or bias forces [1]. Such systems are now termed Brownian ratchets. These are spatially periodic systems in which nonzero current is generated by noise forces of vanishing mean and/or zero-mean deterministic forces. Ratchet-type models have been used in molecular biology in order to explain translocations of motor proteins such as kinesin, dynein, myosin, and their relatives [2]. These enzymes perform practical tasks such as transport of organelles and vesicles, locomotion, and segregation of chromosomes during mitosis. Possible physics applications of ratchet systems, especially for different devices in nano- and microtechnologies, are presently being actively investigated [1,3]. Several mechanisms of noise-driven transport have been proposed [1]: In [4] a spatially periodic potential is switched on and off both deterministically and randomly. This situation is referred to as a flashing ratchet, i.e., it corresponds to a situation with a fluctuating periodic potential [1,5]. Another class of ratchets uses a fluctuating force, which is either of stochastic (i.e., a ''correlation'' ratchet) or deterministic origin (i.e., a ''rocking'' ratchet). In correlation ratchets [6–8], the driving noise is a correlated stochastic force, e.g., the Ornstein-Uhlenbeck process [6,7], an exponentially correlated telegraph signal [6,9], or the kangaroo process [6,8]. Other models use nonequilibrium fluctuations that are modeled by δ-correlated random processes such as white shot noises [10,11] or ratchets driven by pure deterministic noise sources [12,13]. In rocked ratchets [3], the system is subjected to the action of an external, *time-periodic* force and thermal noise. The role of finite inertia and chaotic motion on the ratchet dynamics has been investigated recently for a rocking ratchet in Ref. [12]. In diffusion ratchets [14], the diffusion coefficient of a Fokker-Planck equation is assumed to be a *state-independent* time-periodic function.

In this paper we consider Brownian particles in a spatially periodic potential that are driven by two stochastic processes, namely, Gaussian white noise and white shot noise composed of positively weighted δ pulses that occur at the arrival times of a Poissonian counting process. Such shot noise is abundant in nature: For example, it describes the emission of electrons in diodes, the counting process of emitted photons, and the rate of arriving telephone calls, to name only a few. The former characterizes equilibrium *symmetric* fluctuations in the system at temperature T. The latter models nonequilibrium asymmetric, white fluctuations of zero mean. The *statistical asymmetry* [11] of this shot noise is sufficient to induce directed motion for the particles (finite current); this is so because noise-activated forward and backward transitions then no longer equal each other [10,11]. In Ref. [10] an exact analytical result for the current was derived when the system is at zero temperature $T=0$. Herein we extend this prior study to *finite temperatures* $T>0$. In Sec. II we formulate the model for the ratchet dynamics. A master equation for a probability distribution $P(x,t)$ of the resulting process is a partial integro-differential equation that can be transformed into an equation of continuity expressing the conservation law of probability. This reformulation of the master equation is presented in Sec. III. From the continuity equation for $P(x,t)$ one finds that the probability current $J(x,t)$ is determined, in the stationary regime, by a second-order differential equation. Only for temperature $T=0$ does it reduce to a first-order differential equation, which can be solved analytically for an arbitrary form of the periodic potential $V(x)$ [10]. If $T>0$, the second-order ordinary differential equation for the stationary distribution $P(x)$ with x-dependent coefficients cannot be solved in general. In Sec. IV we discuss the limiting situation with a very strong intensity for the Poissonian fluctuations. In Sec. V we present two asymptotic expansions for the current J with respect to a small ratio ϵ of thermal-noise strength (proportional to temperature) to the shot-noise strength. An equation determining J is a singularly perturbed differential equation because ϵ enters the highest derivative. In Sec. VI we obtain the exact results for a case of a piecewise linear potential. For this stylized situation, we are able to evaluate analytically both the stationary periodic distribution as well as the corresponding current J. In Sec. VII we investigate the dependence of the current on various parameters of the model such as the temperature, the asymmetry of the potential, and the shot noise intensity. In Sec. VIII we also compare the two asymptotic expansions for J versus the exact stationary current. Our findings are summarized in Sec. IX.

II. MODEL FOR RATCHET DYNAMICS

The ratchet-type system studied in the paper is modeled in the presence of strong frictional forces $M\gamma\dot{x}$ (M denotes the mass of the particle and γ is the friction coefficient) by an overdamped stochastic dynamics, i.e.,

$$\dot{x}=f(x)+\xi(t)+\Gamma(t), \tag{1}$$

where

$$f(x)=-\frac{dV(x)}{dx} \tag{2}$$

and $V(x)=V(x+L)$ is a rescaled (divided by $M\gamma$) periodic potential with a spatial period L. The process $\xi(t)$ is white shot noise defined as [15,16]

$$\xi(t)=\sum_{i=1}^{n(t)} z_i\delta(t-t_i)-\lambda\langle z_i\rangle. \tag{3}$$

The Poissonian points t_i are the arrival times of a Poissonian counting process $n(t)$ with parameter λ, i.e., the probability that k delta impulses occur in the interval $(0,t)$ is given by the Poissonian distribution $\Pr\{n(t)=k\}=(\lambda t)^k\exp(-\lambda t)/k!$. Then the distance s between successive Poissonian arrival times $s=t_i-t_{i-1}$ is exponentially distributed with the probability density $\lambda\exp(-\lambda s)$. The parameter λ determines the mean number of the Dirac δ pulses per unit time; it equals the reciprocal of the average sojourn time between two δ kicks. The positive-valued amplitudes $\{z_i\}$ of the δ pulses are random variables independent of each other and of the counting process $n(t)$. The weights $\{z_i\}$ are *exponentially* distributed with the probability density

$$h(z)=A^{-1}\Theta(z)e^{-z/A},\quad A>0, \tag{4}$$

where $\Theta(z)$ is the Heaviside step function. The moments of amplitudes $\{z_i\}$, according to Eq. (4), are given by the relations

$$\langle z_i^k\rangle=k!A^k,\quad k=1,2,3,\ldots. \tag{5}$$

The quantity

$$a=\lambda A=\lambda\langle z_i\rangle \tag{6}$$

describes, according to Eq. (3), the (negative-valued) bias of the shot-noise process realization between consecutive δ pulses.

From Eqs. (3) and (4) it follows that the average and the correlation of noise (3) are given by

$$\langle\xi(t)\rangle=0,\quad\langle\xi(t)\xi(u)\rangle=2D_S\delta(t-u), \tag{7}$$

where the total shot-noise intensity D_S reads

$$D_S\equiv\lambda A^2. \tag{8}$$

The process $\Gamma(t)$ represents thermal fluctuations, i.e., it is Gaussian white noise with

$$\langle\Gamma(t)\rangle=0,\quad\langle\Gamma(t)\Gamma(u)\rangle=2D_T\delta(t-u), \tag{9}$$

where the thermal-noise strength D_T is

$$D_T\equiv Mk_BT/\gamma, \tag{10}$$

with T being temperature of the system. As usual, we assume that $\Gamma(t)$ is not correlated with $\xi(t)$.

A master equation for the probability distribution $P(x,t)$ of the process $x(t)$ defined by Eq. (1) has the form [15,17]

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t}=&-\frac{\partial}{\partial x}[f(x)-\lambda A]P(x,t)\\&+\lambda\int_{-\infty}^{\infty}h(z)[P(x-z,t)-P(x,t)]dz\\&+D_T\frac{\partial^2}{\partial x^2}P(x,t).\end{aligned} \tag{11}$$

The right-hand side of this equation consists of three parts: The first term denotes the drift, including a Poissonian-noise-induced part proportional to $\lambda A=\lambda\langle z_i\rangle$; the second term is related to the Poissonian process; and the third term corresponds to the thermal diffusion process. With nonzero thermal noise the stationary probability has a support over the whole x axis. Moreover, with the drift part being periodic, the stationary probability for the one-dimensional Markov process depends on the choice of the boundary conditions (BC's). With two reflecting BC's at $x_l=x$ and $x_r=x+L$, the probability current is zero; in contrast, a finite, stationary probability current emerges if periodic BC's are chosen.

III. CONTINUITY EQUATION FOR PROBABILITY AND EQUATION FOR PARTICLE CURRENT

For studying transport characteristics of process (1), it is useful to transform Eq. (11) to the continuity equation

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}J(x,t). \tag{12}$$

This conservation law defines the probability current $J(x,t)$. To obtain this current, let us introduce the shift operator by the relation

$$\exp(-z\partial/\partial x)P(x,t)=P(x-z,t). \tag{13}$$

Next we expand the shifted probability into a Taylor series. After integration of the second line in Eq. (11) we can, upon observation of the exponential form of the density $h(z)$ for the weights in Eq. (4) and the moments in Eq. (5), recast the master equation in Eq. (11) into the form of a continuity equation. In doing so, we use the identity

$$\sum_{n=0}^{\infty}(-A)^n\frac{\partial^n}{\partial x^n}P(x,t)$$
$$=A^{-1}\int_0^{\infty}dye^{-y/A}\exp[-y\partial/\partial x]P(x,t)$$
$$=\int_0^{\infty}dyA^{-1}e^{-y/A}P(x-y,t)$$
$$=\int_{-\infty}^{\infty}h(x-z)P(z,t)dz. \tag{14}$$

With this identity and the relation $A\partial h(x-z)/\partial x=-h(x-z)$ the probability current $J(x,t)$ can be expressed as

$$J(x,t)=f(x)P(x,t)-D_S\frac{\partial}{\partial x}\int_{-\infty}^{\infty}h(x-z)P(z,t)dz$$
$$-D_T\frac{\partial}{\partial x}P(x,t). \tag{15}$$

Let us notice that Eq. (12) with Eq. (15) can be interpreted as a *spatially nonlocal diffusion* equation, i.e.,

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}f(x)P(x,t)+\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\mathcal{D}(x,z)P(z,t)dz, \tag{16}$$

with an effective diffusion function

$$\mathcal{D}(x,z)=D_Sh(x-z)+D_T\delta(x-z). \tag{17}$$

It consists of nonlocal (Poissonian) and local (thermal) parts. In the limiting case

$$\lambda\to\infty,\quad A\to 0,\quad D_S=\lambda A^2=\text{const} \tag{18}$$

the nonlocal part tends to a local diffusion function $D_S\delta(x-z)$. In this limit, $a\to\infty$ and Poissonian white shot noise tends to Gaussian white noise with the diffusion strength D_S.

The solution $P(x,t)$ of Eq. (16) is a periodic function of x, i.e., $P(x+L,t)=P(x,t)$, if we choose an initial distribution $P(x,0)$ that is periodic with respect to x. Combining Eqs. (1) and (12) with Eq. (16) thus yields the relation between the average of the particle velocity $\langle v(t)\rangle$ and the current $J(x,t)$, namely,

$$\langle v(t)\rangle=\langle\dot{x}\rangle=\langle f(x)\rangle=\int_c^{c+L}J(x,t)dx, \tag{19}$$

which is valid for any real number c.

The stationary current J follows from Eq. (15) in the long-time limit $t\to\infty$ with periodic boundary conditions imposed on the corresponding stationary probability $P(x)=P(x+L)$. With $h(z)$ given in Eq. (4), the current J is determined by an *ordinary differential equation of second order*, namely,

$$-D_TAP''(x)-[D_T+D_S-Af(x)]P'(x)$$
$$+[f(x)+Af'(x)]P(x)=J. \tag{20}$$

Here and below the prime denotes a derivative with respect to the argument of a function. This equation can be integrated for temperature $T=0$, i.e., when $D_T=0$. In this case the current J is obtained in closed form in terms of two quadratures [10]. For $T>0$, arbitrary values for the noise parameters, and a general form for the periodic potential $V(x)$, Eq. (20) cannot be solved explicitly in closed form.

IV. ASYMPTOTIC REGIME OF LARGE SHOT-NOISE INTENSITY

Let us consider the case with a finite thermal temperature T. Then there is only one nontrivial limiting case of Poissonian white fluctuations for which the current can be evaluated analytically. This is the situation when

$$\lambda\to 0,\quad A\to\infty,\quad \lambda A\equiv a\ \text{(fixed)}, \tag{21}$$

where a, by virtue of Eq. (6), characterizes the negative base value of the white-shot-noise realization $\xi(t)$. The above limit means that the strength $D_S=\lambda A^2$ of the Poissonian white noise tends to infinity, while its value between δ kicks is fixed at $-a$. In the regime of very large values of D_S, the stationary distribution $\mathcal{P}(x)$ is determined by the differential equation [cf. Eq. (20)]

$$D_T\frac{d^2}{dx^2}\mathcal{P}(x)+\frac{d}{dx}[a-f(x)]\mathcal{P}(x)=0, \tag{22}$$

with two imposed conditions: periodicity and normalization of $\mathcal{P}(x)$ to unity within a spatial period L of the potential $V(x)$. The resulting periodic solution is found to read

$$\mathcal{P}(x)=\frac{1}{W}e^{-\phi(x)}\int_x^{x+L}e^{\phi(z)}dz, \tag{23}$$

where W takes care of the normalization of $\mathcal{P}(x)$,

$$W=\int_0^L e^{-\phi(x)}\int_x^{x+L}e^{\phi(z)}dz\,dx, \tag{24}$$

and the generalized potential reads

$$\phi(x)=[ax+V(x)]/D_T. \tag{25}$$

Because $\phi(x)$ is not periodic, the difference $\phi(x+L)-\phi(x)\neq 0$. Hence the generalized potential possesses a slope (an average bias) and thus supports a nonzero stationary current. Its value follows from Eq. (15) as [cf. Eq. (19)],

$$J = L^{-1}\langle f(x)\rangle = L^{-1}\int_c^{c+L} f(x)\mathcal{P}(x)dx. \tag{26}$$

If the temperature T of the system tends to zero, i.e., $D_T\to 0$, we find from Eq. (22) in this asymptotic regime (we recall here that the case with zero temperature can be solved analytically in terms of two quadratures; see [10]) for the periodic probability

$$[a-f(x)]\mathcal{P}(x) = C_0, \tag{27}$$

where C_0 is a constant. Integrating this equation over the interval $[c,c+L]$ yields

$$J = \frac{a}{L} - C_0. \tag{28}$$

The first term a/L is due to forward transitions generated by δ kicks, i.e., the first term in Eq. (3). The second term C_0 is due to backward transitions. If $f(x)<a$ for all x, then C_0 is different from zero, yielding

$$\mathcal{P}(x) = \frac{C_0}{a-f(x)} \quad \text{with} \quad C_0^{-1} = \int_c^{c+L}\frac{dx}{a-f(x)}. \tag{29}$$

In this case, Eq. (26) reduces to Eq. (14) in Ref. [10]. On the other hand, if there is a subinterval of $[c,c+L]$ for which $f(x)>a$, shot-noise-activated backward transitions are impossible. Therefore, we have $C_0=0$. This result follows also from Eq. (27): Because $\mathcal{P}(x)\geqslant 0$ for any x and $a-f(x)$ changes sign while x changes in $[c,c+L]$, Eq. (27) can be fulfilled only when $C_0\equiv 0$, yielding for the current $J=a/L$.

In the Gaussian white-noise limit (18) or in the limit $D_S\to 0$, the system is driven solely by Gaussian white noise; consequently, the current J vanishes identically. In the limit $a\to\infty$ and $D_S\to\infty$, i.e., when $\lambda\to\infty$ and A is fixed, or when λ is fixed and $A\to\infty$, Eq. (20) reduces to $P'(x)=0$. Thus the probability density is $P(x)=L^{-1}$, and as a consequence $J\to 0$; see Eq. (26).

V. ASYMPTOTIC EXPANSIONS

The previously exactly solved case for $T=0$ [10] suggests that, instead of solving Eq. (20) directly, which, in general, is not possible, one can attempt to determine an approximate solution of Eq. (20) for small temperature by use of perturbation techniques. To this aim, it is desirable to transform Eq. (20) into a dimensionless form, i.e.,

$$-\epsilon p''(y) - [\epsilon + 1 - \tilde{f}(y)]p'(y) + [\tilde{f}(y) + \tilde{f}'(y)]p(y) = j, \tag{30}$$

where the rescaled quantities are defined by the relations

$$y = \frac{x}{A}, \quad j = \frac{J}{\lambda}, \quad p(y) = AP(Ay), \quad \tilde{f}(y) = \frac{f(Ay)}{\lambda A}. \tag{31}$$

The non-negative parameter ϵ is a ratio of the thermal noise intensity to the intensity of shot noise, namely,

$$\epsilon = \frac{D_T}{D_S}. \tag{32}$$

Next we treat Eq. (30) as a perturbed differential equation with a small parameter ϵ. It is a singularly perturbed equation because ϵ occurs at the highest (second-order) derivative. A great deal of work in singular perturbations has been devoted to boundary problems, as well as to initial-value problems [18,19]. Our problem (30) belongs neither to the former nor to the latter. In a perturbation problem an approximation is sought by solving the reduced problem obtained from the original one by setting $\epsilon=0$. The solution of the reduced problem in general is not an approximation to the exact solution of the full problem on the whole intervals of independent variables and parameters of the system. This is why it is rather difficult to obtain a perturbation expansion of a solution to Eq. (30) that is uniformly convergent. What can be done is to construct an asymptotic solution as a formal series using *ad hoc* arguments, but without proving the validity of it and without error estimation of approximations. Unfortunately, there is no systematic and unique approach for constructing asymptotic expansions. Here we present two such formal expansions.

A. Regular expansion

The first type of the expansion is an ansatz of the form

$$p^{(1)}(y) = p_0(y) + \sum_{n=1}^{\infty}\epsilon^n p_n(y), \quad j^{(1)} = j_0 + \sum_{n=1}^{\infty}\epsilon^n j_n, \tag{33}$$

where $\{p_0(y),j_0\}$ is a solution of the truncated system (when $\epsilon\equiv 0$). For convenience, we call Eq. (33) a *regular ϵ expansion*. In this approach, two terms of Eq. (30), namely, $\epsilon p''(y)$ and $\epsilon p'(y)$, are treated as a perturbation. Substituting Eq. (33) into Eq. (30) and equating coefficients of equal power in ϵ, we obtain equations determining successively $p_n(y)$ and j_n. They have the form

$$-D_0(y)p_0'(y) + F(y)p_0(y) = j_0,$$

$$-D_0(y)p_n'(y) + F(y)p_n(y) = j_n + G_{n-1}(y),$$

$$n = 1,2,3,\ \dots\ . \tag{34}$$

The functions $D_0(y)$, $F(y)$, and $G_n(y)$ are given by

$$D_0(y) = 1 - \tilde{f}(y), \quad F(y) = \tilde{f}(y) + \tilde{f}'(y),$$

$$G_n(y) = p_n''(y) + p_n'(y), \quad n = 0,1,2,\ \dots . \tag{35}$$

The set of equations (34) is supplemented with the periodicity conditions

$$p_n(y+l) = p_n(y), \quad l = L/A, \quad n = 0,1,2,\ \dots . \tag{36}$$

The normalization of the distribution $p(y)$ over a rescaled period l leads to the additional conditions

$$\int_0^l p_n(y)dy = \delta_{0n}, \quad n = 0,1,2,\ \dots . \tag{37}$$

Now the problem (34)–(37) can be solved and the zeroth-order contribution emerges as

$$p_0(y)=\frac{1}{Q}e^{-\Psi(y)}\int_y^{y+l}D_0^{-1}(z)e^{\Psi(z)}dz, \tag{38}$$

where

$$Q=\int_0^l e^{-\Psi(y)}dy\int_y^{y+l}D_0^{-1}(z)e^{\Psi(z)}dz \tag{39}$$

takes into account normalization of the probability distribution and

$$\Psi(y)=-\int_0^y \frac{F(z)}{D_0(z)}dz \tag{40}$$

is a generalized potential of the unperturbed system. The zeroth-order approximation j_0 to the current j reads [10]

$$j_0=\frac{1}{Q}[1-e^{\Psi(l)}]. \tag{41}$$

It is a solution of the truncated system corresponding to the case when $\Gamma(t)\equiv 0$ in Eq. (1). Then the system is driven by white shot noise only. This case has been analyzed in detail analytically in Refs. [10,11]. It is assumed that the *unperturbed diffusion function* obeys $D_0(y)>0$. Then the current of the unperturbed system at temperature $T=0$ occurs nontrivially [10,11].

The higher-order contributions to $p(y)$ have the form

$$p_n(y)=\frac{e^{-\Psi(y)}}{1-e^{\Psi(l)}}\left\{j_n\int_y^{y+l}D_0^{-1}(z)e^{\Psi(z)}dz + \int_y^{y+l}G_{n-1}(z)D_0^{-1}(z)e^{\Psi(z)}dz\right\},$$

$$n=1,2,3,\ \ldots, \tag{42}$$

where the higher-order terms j_n of the total current j are determined by the relations

$$j_n=-\frac{1}{Q}\int_0^l e^{-\Psi(y)}\int_y^{y+l}G_{n-1}(z)D_0^{-1}(z)e^{\Psi(z)}dz\,dy,$$

$$n=1,2,3,\ \ldots\ . \tag{43}$$

Both $p_n(y)$ and j_n depend on lower-order contributions via the functions $G_{n-1}(y)$ expressed by $p''_{n-1}(y)$ and $p'_{n-1}(y)$; cf. (35).

B. Renormalized expansion

The second type of the expansion is postulated in the form

$$p^{(2)}(y)=\sum_{n=0}^{\infty}\epsilon^n p_n(y,\epsilon), \quad j^{(2)}=\sum_{n=0}^{\infty}\epsilon^n j_n(\epsilon), \tag{44}$$

where the unknown functions are solutions to the set of differential equations

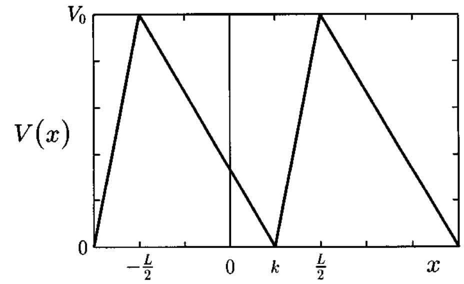

FIG. 1. Spatially periodic sawtooth potential $V(x)$ of period L, barrier height V_0, and asymmetry parameter k.

$$-D(y,\epsilon)p'_0(y,\epsilon)+F(y)p_0(y,\epsilon)=j_0(\epsilon),$$

$$-D(y,\epsilon)p'_n(y,\epsilon)+F(y)p_n(y,\epsilon)=j_n(\epsilon)+p''_{n-1}(y,\epsilon),$$

$$n=1,2,3,\ \ldots, \tag{45}$$

subject to the periodicity condition (36) and the normalization condition in Eq. (37).

The function $D(y,\epsilon)$ has the form

$$D(y,\epsilon)=\epsilon+1-\tilde{f}(y) \tag{46}$$

and may be called a *renormalized diffusion function*: A correction related to thermal noise is included in the shot-noise diffusion function. In the second approach, only one term of Eq. (30), i.e., $\epsilon p''(y)$, is treated as a small perturbation. On the contrary, the term $\epsilon p'(y)$ is treated exactly and enters the unperturbed diffusion function $D_0=D(y,\epsilon=0)$, renormalizing it.

The solution of Eq. (44) has a similar form given to that in Eqs. (38)–(43), but with corresponding changes $D_0(y)\to D(y,\epsilon)$ and $G_n(y)\to p''_n(y,\epsilon)$ in all formulas (38)–(43). A domain of validity of these two asymptotic expansions is strongly restricted by the requirement that the diffusion functions are positive: In nonrescaled variables, $D_S-Af(x)>0$ for the regular expansion and $D_T+D_S-Af(x)>0$ for the renormalized expansion. A comparison with exact results is discussed in Sec. VIII for a particular form of the potential $V(x)$.

VI. EXACT SOLUTION FOR THE SAWTOOTH POTENTIAL

The problem (20) can be solved analytically for special forms of the potential $V(x)$ only. As an example, we present analysis of the system (1) for a piecewise linear, sawtooth-like potential (see Fig. 1)

$$V(x)=\begin{cases} -\dfrac{2V_0}{L+2k}(x-k), & x\in[-L/2,k]\mathrm{mod}L \\ \dfrac{2V_0}{L-2k}(x-k), & x\in[k,L/2]\mathrm{mod}L, \end{cases} \tag{47}$$

where $V_0>0$ and $k\in(-L/2,L/2)$ determines the asymmetry of the potential: For $k=0$ it is reflection symmetric; for $k\neq 0$ the reflection symmetry of $V(x)$ is broken.

Because the force (2) is periodic, the stationary distribution $P(x)$ being a solution of Eq. (20) is periodic and it is sufficient to consider the problem (20) on the interval

$[x_0, x_0+L]$ for a fixed $x_0 \in (-L/2, k)$. It ensures the smoothness of solutions at the boundaries of the interval. The corresponding force $f(x)$ has the form

$$f(x) = \frac{2V_0}{L+2k}\Theta(x+L/2)\Theta(k-x) - \frac{2V_0}{L-2k}\Theta(x-k)\Theta(L/2-x) + \frac{2V_0}{L+2k}\Theta(x-L/2)\Theta(L/2+k-x) \tag{48}$$

for any $x \in [x_0, x_0+L]$. This form of the force suggests the ansatz

$$P(x) = p_1(x)\Theta(x+L/2)\Theta(k-x) + p_2(x)\Theta(x-k) \times\Theta(L/2-x) + p_3(x)\Theta(x-L/2)\Theta(L/2+k-x) \tag{49}$$

for the solution of Eq. (20). In this equation, the relation

$$p_3(x) = p_1(x-L)$$

is fulfilled, due to the periodicity condition. Substituting Eq. (49) into Eq. (20) leads to an equation of the structure

$$g(x) + \alpha_1\delta(x-k) + \alpha_2\delta(x-L/2) + \beta_1\delta'(x-k) + \beta_2\delta'(x-L/2) = 0, \tag{50}$$

with a given function $g(x)$ and some constant coefficients α_i and β_i $(i=1,2)$. From the Lemma 3.1.2 in [20] it follows that this equation holds when

$$g(x) = 0, \quad \alpha_i = 0, \quad \beta_i = 0, \quad i = 1,2. \tag{51}$$

From the first equation of Eqs. (51) one obtains equations for the functions $p_i(x)$ $(i=1,2)$ in the form

$$-D_T A p_1''(x) - \left[D_S + D_T - \frac{2AV_0}{L+2k}\right]p_1'(x) + \frac{2V_0}{L+2k}p_1(x) = J,$$
$$-D_T A p_2''(x) - \left[D_S + D_T + \frac{2AV_0}{L-2k}\right]p_2'(x) - \frac{2V_0}{L-2k}p_2(x) = J. \tag{52}$$

Boundary conditions for these ordinary differential equations follow from the remaining equations of Eqs. (51) and read

$$p_1(k) = p_2(k), \tag{53}$$

$$p_1(-L/2) = p_2(L/2), \tag{54}$$

$$D_T[p_1'(k) - p_2'(k)] = \frac{4V_0 L}{(L+2k)(L-2k)}p_1(k), \tag{55}$$

$$D_T[p_1'(-L/2) - p_2'(L/2)] = \frac{4V_0 L}{(L+2k)(L-2k)}p_1(-L/2). \tag{56}$$

Normalization of $P(x)$ leads to the following fifth condition, namely,

$$-D_T A(L/2+k)[p_1'(-L/2) - p_1'(k)] + D_T A(L/2-k) \times[p_2'(L/2) - p_2'(k)] + \frac{L}{2}(D_S + D_T)[p_2(L/2) - p_2(k) - p_1(k) + p_1(-L/2)] + J[(L/2-k)^2 - (L/2+k)^2] = -V_0. \tag{57}$$

A solution of Eq. (52) has the form (for a similar solution technique with the white shot noise substituted by a dichotomous two-state process see Ref. [5])

$$p_1(x) = B_1 e^{w_{11}x} + B_2 e^{w_{12}x} + J(L+2k)/2V_0, \tag{58}$$

$$p_2(x) = C_1 e^{w_{21}x} + C_2 e^{w_{22}x} - J(L-2k)/2V_0, \tag{59}$$

where

$$w_{11} = \frac{-\Omega_- - \sqrt{\Omega_-^2 + 8D_T A V_0/(L+2k)}}{2D_T A}, \tag{60}$$

$$w_{12} = \frac{-\Omega_- + \sqrt{\Omega_-^2 + 8D_T A V_0/(L+2k)}}{2D_T A}, \tag{61}$$

$$w_{21} = \frac{-\Omega_+ - \sqrt{\Omega_+^2 - 8D_T A V_0/(L-2k)}}{2D_T A}, \tag{62}$$

$$w_{22} = \frac{-\Omega_+ + \sqrt{\Omega_+^2 - 8D_T A V_0/(L-2k)}}{2D_T A}, \tag{63}$$

$$\Omega_\mp = D_S + D_T \mp \frac{2AV_0}{L \pm 2k}. \tag{64}$$

The current J and the four constants B_i, C_i $(i=1,2)$ are determined by five conditions (53)–(57). This yields a nonhomogeneous system of five linear algebraic equations. Hence the problem is solved and evaluation of the current is a matter of linear algebra. Because J is a quotient of two determinants of the fifth degree, the explicit form of J emerges as a complex expression, which is not reproduced here. The analysis of the current with its corresponding graphical representation is the subject of the next section.

VII. DISCUSSION OF RESULTS

In this section we analyze transport properties in the piecewise linear potential (47). There are six parameters in our ratchet model, namely, D_T, the thermal-noise strength proportional to temperature of the system; (D_S, a), which characterize the Poissonian δ-correlated fluctuations; and (V_0, k, L), which determine the potential $V(x)$. A general note concerns the sign of the current: J is positive for any (nonzero, finite) values of parameters. This is so because of the positive-valued weights of the δ kicks. The current is a monotonically increasing function of the shot-noise intensity D_S. Starting from zero at $D_S = 0$, it saturates to the maximal

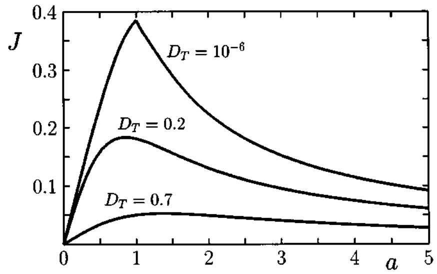

FIG. 2. Current J vs parameter a [which measures the negative base value of the white-shot-noise realization $\xi(t)$ between the positive-valued δ kicks] for various thermal-noise intensities D_T, fixed white-shot-noise intensity $D_S=1$, in the symmetric ($k=0$), periodic sawtooth potential with period $L=2$ and barrier height $V_0=1$.

value given by Eq. (26). Qualitatively, the same property has been observed for zero thermal temperature $T=0$ [10].

A. Current versus shot-noise bias level

The dependence of J versus the shot-noise bias level a in Eq. (6) is visualized in Fig. 2. Our figures have been obtained by solving the above-mentioned system of five linearly coupled algebraic equations by straightforward, simple numerical means. There is an optimal value $a_{\max}$ for which the current is maximal. As the temperature of the system decreases to zero, $a_{\max}$ approaches the value $2V_0/(L+2k)$ from below. If $a>2V_0/(L+2k)$ at $T=0$, both backward and forward transitions take place for the Brownian dynamics of the particles. Throughout our discussion here, we evaluate the analytic results for $T=0$ by using, for convenience, in our program a very small temperature of $D_T=10^{-6}$. On the contrary, if $a<2V_0/(L+2k)$ at $T=0$, only forward transitions drive the particle [11]. If $T>0$, this no longer holds true. For any a, there are now both backward and forward transitions being induced by nonzero Gaussian white noise.

B. Current versus potential asymmetry

The current depends strongly on the asymmetry parameter k of the potential. We define the asymmetry as being *positive* if $k<0$ and vice versa. The case $k=0$ corresponds to a symmetric periodic potential. Positive asymmetry means that when starting from minima of the potential its slope in the x-increasing (right) direction is less than in the x-decreasing (left) direction or, put differently, climbing the barrier is easier towards the right than towards the left. In Figs. 3(a)–3(c) we depict the current dependence upon the asymmetry parameter k of the potential. In a ''hot'' system, i.e., when the temperature T is relatively high, the symmetric thermal noise dominates so that the current is ruled increasingly by an equilibrium dynamics. Hence the current tends to zero independently of the specific form of the ratchet potential, which implies that the current is almost symmetric with respect to asymmetry; see Figs. 3(b) and 3(c). Moreover, we note that, at fixed temperature T, the current increases with increasing $|k|$.

At $T=0$, as well as at low temperatures and for strong intensity D_S of Poissonian fluctuations, new effects arise; see the cases $D_S=1$ and $D_S=0.5$ in Fig. 3(a). As k changes from a maximal positive asymmetry at $k=-L/2$, the current grows to a (local or global) value. Next J diminishes, attaining a (global or local) minimal value. The minimum is not for a symmetric potential at $k=0$, but is shifted towards a positive-k value, i.e., a *negative* asymmetry for the ratchet potential. A further increase of asymmetry leads to an increase of the current. This behavior is most pronounced at $T=0$ and can be explained as follows: If $k<V_0/a-L/2$, there is no net flux in the left direction. Strong shot-noise intensity D_S means that δ pulses are rare (λ is small) and amplitudes z_i in Eq. (3) are large (but $\lambda A=a=$const). Between δ pulses and for k close to $-L/2$, particles are localized near a minimum of the potential $V(x)$ since the deterministic relaxation times from the left and from the right maxima to the minimum are shorter than $1/\lambda$. If k increases then the distance between a minimum and the neighboring

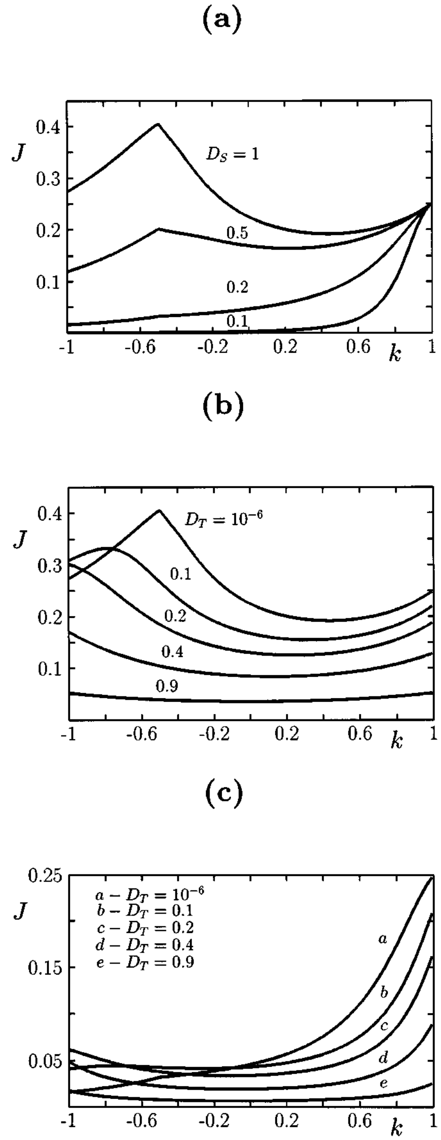

FIG. 3. Current J shown vs asymmetry parameter k of the sawtooth potential (47) for fixed $a=2$, $L=2$, and $V_0=1$, for (a) $D_T=10^{-6}$ and various values of shot-noise strength D_S, (b) $D_S=1$ and various values of thermal-noise strength D_T, and (c) $D_S=0.2$ and various values of thermal-noise strength D_T.

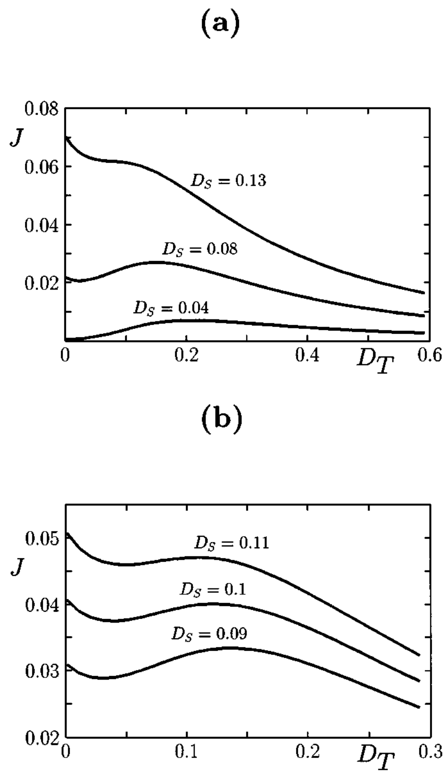

FIG. 4. Plot of the current J as a function of thermal-noise intensity D_T (or rescaled temperature T) for fixed $a=0.5$, $L=2$, $V_0=1$, $k=0$, and selected values of white-shot-noise strength D_S. (a) three distinctive (monotonic decrease, minimum-maximum, and bell-shaped) behaviors of $J(D_T)$ are depicted. (b) depicts how a global maximum for J changes into a local maximum while increasing the white-shot-noise intensity D_S.

right maximum becomes shorter. Smaller and more probable amplitudes [cf. the distribution (4)] are able to displace particles over the maximum into the right direction. This feature holds with $k<0$ up to $k=V_0/a-L/2$ at $T=0$. When $k>V_0/a-L/2$, a flux arises also towards the left direction and grows as k increases. As a consequence, the total current in the right direction decreases as k increases; cf. in Fig. 3(a) for $D_S=1$ and $k>-0.5$. Starting from a certain positive-k value (*negative asymmetry*) the growth of k causes an increase of the current up to the maximal value

$$J=\frac{v_r}{L}=\frac{2V_0}{(L+2k)L}\to\frac{V_0}{L^2}\quad\text{as}\quad k\to L/2. \tag{65}$$

Here v_r is the deterministic part of the particle velocity in the right direction. This is so because the probability that, after a δ kick, the particle falls into the interval $(k,L/2)$ is much less compared to the case where it falls into the interval $(-L/2,k)$.

C. Current versus temperature

The stationary current J as a function of the thermal noise intensity D_T (or rescaled temperature T of the system) is plotted in Figs. 4(a) and 4(b) for several values of the shot-noise strength D_S and fixed remaining parameters. One can observe three radically different types of the current behavior as the control parameter D_S is varied.

(i) The current is a monotonically decreasing function of temperature. (ii) As temperature increases from zero, the current starts from a nonzero value, decreasing to a local minimum; next it grows, attaining a (local or global) maximum, and then J approaches zero as $T\to\infty$. (iii) Increasing the temperature from zero results first in a rise of the current and then its fall. There is one unique temperature maximizing the current. Put differently, the current J exhibits a bell-shaped behavior versus increasing temperature.

VIII. APPROXIMATE SOLUTIONS

For a comparison between the two asymptotic expansions presented in Sec. V with the exact results obtained for the case of the piecewise linear potential (47), we truncate the series (33) and (44) and restrict ourselves to terms of first order with respect to the expansion parameter ϵ. Within the original, i.e., nonrescaled variables, indicated by J, and for the potential barrier V_0, the renormalized expansion, denoted by the superscript 2 yields

$$J^{(2)}\approx J_0(D_T)+\frac{D_TD_S}{a}J_1(D_T), \tag{66}$$

where

$$J_0(D_T)=\frac{1-e^{(L/2-k)r-(L/2+k)s}}{N} \tag{67}$$

and

$$\begin{aligned}J_1(D_T)=&\frac{(L/2-k)^2r(r+s)}{sV_0^2N^2}[e^{(L/2-k)r}-e^{(L/2-k)r-(L/2+k)s}]\\&\times[-1+2r(1-e^{-(L/2+k)s})-e^{-(L/2+k)s}\\&+e^{-(L/2-k)r}+e^{(L/2-k)r-(L/2+k)s}]\\&+\frac{(L/2+k)^2s(r+s)}{rV_0^2N^2}[e^{(L/2-k)r-(L/2+k)s}\\&-e^{-(L/2+k)s}][-1-2s(1-e^{(L/2-k)r})-e^{(L/2-k)r}\\&+e^{(L/2+k)s}+e^{(L/2-k)r-(L/2+k)s}].\end{aligned} \tag{68}$$

The normalization constant reads

$$\begin{aligned}N\equiv N(D_T)=&\frac{L^2}{V_0^2}(D_T+D_S)[e^{(L/2-k)r}+e^{-(L/2+k)s}\\&-e^{(L/2-k)r-(L/2+k)s}-1]\\&+2k\frac{L}{V_0}[1-e^{(L/2-k)r-(L/2+k)s}].\end{aligned} \tag{69}$$

The quantities r and s are given by

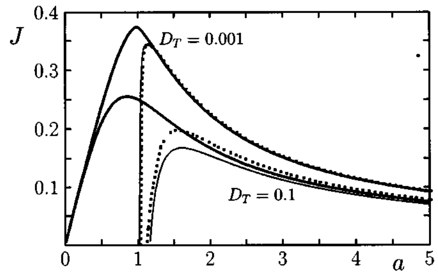

FIG. 5. Exact results (thick solid line) are compared against two approximations obtained from the the first-order asymptotic expansions: regular (thin solid line) and renormalized (dotted line) expansions. The parameters are $D_S=1$, $L=2$, $V_0=1$, and $k=0$ for two values of the expansion parameter $\epsilon=D_T/D_S=D_T=0.001$ and 0.1.

$$r\equiv r(D_T)=\left[\left(\frac{L}{2}-k\right)\frac{D_T+D_S}{V_0}+\frac{D_S}{a}\right]^{-1}, \tag{70}$$

$$s\equiv s(D_T)=\left[\left(\frac{L}{2}+k\right)\frac{D_T+D_S}{V_0}-\frac{D_S}{a}\right]^{-1}. \tag{71}$$

The regular expansion, denoted by the superscript 1, leads to the relation

$$J^{(1)}\approx J_0(0)+\frac{D_T D_S}{a}J_1(0). \tag{72}$$

As we mentioned before, the expansions can be used only when the unperturbed diffusion function in Eq. (35) and the renormalized diffusion function in Eq. (46) are positive for any x. For the sawtooth potential (47), this implies the conditions

$$a>a_1=2/(k+L/2), \quad a>a_2=2/[(k+L/2)(1+D_T/D_S)] \tag{73}$$

for the unperturbed and renormalized diffusion functions, respectively. As depicted in Fig. 5, the regime of validity of these two approximations is governed not only by the parameter ϵ defined in Eq. (32) but also by the base level a of white shot noise $\xi(t)$, which is restricted by Eq. (73). If ϵ is rather small (top two lines in Fig. 5), the two expansions reproduce the exact result for a decreasing from infinity up to $a\approx a_1$. With increasing ϵ, the deviations from the exact result become more pronounced. Note, however, that the renormalized expansion (dotted lines) does provide better agreement.

IX. SUMMARY

We have studied properties of the current in periodic structures generated by white shot noise and driven by thermal fluctuations. The stationary current is determined by the ordinary differential equation of second order; see Eq. (20). Because this equation cannot be solved by analytical means we have considered various specific situations. For an arbitrary form of the spatially periodic potential, the current can be analytically described in closed form in the asymptotic regime of very strong intensity for Poissonian δ-correlated fluctuations (Sec. IV). Two asymptotic expansions are constructed in Sec. V. The expansion parameter is defined as a quotient of the Gaussian white-noise intensity and the white-shot-noise intensity. These expansions are not uniformly convergent and their domains of validity are determined not only by the expansion parameter, but also by the remaining parameters of the stochastic dynamics.

Exact analytical results are obtained for the piecewise linear sawtoothlike potential (47). The most interesting findings are visualized in the figures. The current vs temperature characteristics are strongly influenced by other parameters of the model, such as the intensity D_S of Poissonian white fluctuations. A notable feature of our ratchet model is the occurrence, for specific choices of the parameter set, of *two* characteristic temperatures at which the current is locally minimal and/or (locally or globally) maximal. The fact that there exists a single characteristic temperature that maximizes the current has been emphasized in [3]. Nevertheless, we believe that the existence of a temperature that locally *minimizes* the current can have practical consequences too. Such a temperature-induced minimum can play a useful role in the design of devices that separate particles. For example, the simultaneous presence of a minimum and a maximum can be of use to separate, with *minimal dispersion* in velocity, two classes of particles. The corresponding regimes are controlled by the value of the thermal noise intensity D_T, which is governed by mass and friction strength parameters; cf. Eq. (10). Suitable systems where these ideas may apply are shot-noise-driven transport in periodic superlattices or biological motor proteins that move along periodic track filaments where shot noise mimics the nonequilibrium stochastic hydrolysis of adenosine triphosphate.

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